

Refutation of ten ZFC axioms from Tiles book on set theory as Cantor's paradise

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·, ∘ , ⊗ ; \ Not And ;
> Imply, greater than, →, ⇒ , ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ←, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠, ⊕ ;
% possibility, for one or some, ∃, ∃!, ◊, M ; # necessity, for every or all, ∀, □, L ;
(z=z) T as tautology, ⊤, ordinal 3 ; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero ;
(%z>#z) N as non-contingency, Δ, ordinal 1 ; (%z<#z) C as contingency, ∇, ordinal 2 ;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Notes: for clarity, we usually distribute quantifiers onto each designated variable; and for ordinal arithmetic, the result is implied.

From: Tiles, M. (1989). The philosophy of set theory: An historical introduction to Cantor's paradise. Dover Publications. ISBN-13: 978-0-486-43520-6. $15.95.

[See the quoted text below at this third page.]

Axiom of extensionality:

LET p, q, r, s: x, y, z, w.

((#r<#p)=(#r<#q))>(#p=#q) ; TCCT TTTT TCCT TTTT (6.2.1.2)

Null set axiom:

~(#q<%p) = (s=s) ; TTCT TTCT TTCT TTCT (6.2.2.2)

Pair set axiom:

((#s<%r)=(#s=#p))+(#s=#q) ; TTCT TTCT CTTT CTTT (6.2.3.2)

Sum set axiom:

LET p, q, r, s: a, y, z, w.
(#r<q)=((%s<p)&(#r<%s)) ; TTTT CCTT TTTT CCTT (6.2.4.2)

Axiom of infinity:

LET p, q, r, s: x, y, z, s or w.

((s@s)<%p) & ((#q<%p)>((%r<%p)&(((#s<%r)=(#s<#q))+(#s=#q)))) ;
FFFF FFFF FFFF FFFF (6.2.5.2)

Axiom of foundation:

```
~(#p=(s@s))>((%q<#p)&((#r<#p)>~(#r<%q))) ;
                          TCTC TCTC TCTC TCTC                    (6.2.6.2)
```

Subset axiom:

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(#r<%q)=((#r<#p)&(s&#r)) ; TTTT CCTT TTTT TCCT                   (6.2.7a.2)
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Replacement axiom:

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LET    s, t, u, v, w, x, y, z:     F, t, u, v, w, x, y, z.

((y<x)>((s&(y&z))&((s&(y&w))>(w=z)))) > ((#u<%v)=((%t<#x)&(s&(%t&#u)))) ;
                          TTTT TTTT TTTT TTTT} 2} 2} 1
                          TTTT TTTT CCCC CCCC} 1}   }
                          TTTT TTTT TTTT TTTT} 4}   }
                          TTTT TTTT CCCC CCCC} 1}   }
                          TTTT TTTT TTTT TTTT} 2} 2}
                          CCCC CCCC CCCC CCCC} 2}   }
                          TTTT TTTT TTTT TTTT} 4}   }
                          TTTT TTTT TTTT TTTT}16} 1}
                          TTTT TTTT TTTT TTTT} 2} 2}
                          CCCC CCCC CCCC CCCC} 2}   }
                          TTTT TTTT TTTT TTTT} 4}   }
                          TTTT TTTT TTTT TTTT} 2} 2} 2
                          TTTT TTTT CCCC CCCC} 1}   }
                          TTTT TTTT TTTT TTTT} 4}   }
                          TTTT TTTT CCCC CCCC} 1}   }
                          TTTT TTTT TTTT TTTT} 2} 2}
                          CCCC CCCC CCCC CCCC} 2}   }
                          TTTT TTTT TTTT TTTT} 4}   }          (6.2.7b.2)
```

Power set axiom:

```
(#z<%y)=((#w<#z)>(#w<#x)) ;
                          FFFF FFFF FFFF FFFF}24} 2
                          NNNN NNNN NNNN NNNN} 8}
                          NNNN NNNN NNNN NNNN}32} 1
                          FFFF FFFF FFFF FFFF}32}                (6.2.8.2)
```

Axiom of choice:

```
(((#y<#x)>~(#y=(s@s)))&(((#y<#x)&(#z<#x))&((#y=#z)>~((%w<#y)&(%w<#z)))))
> ((#y<#x)>(((%z<%u)&(%z<#y))&(((#w<%u)&(#w<#y))>(#w=%z)))) ;
                          TTTT TTTT TTTT TTTT}96
                          CCCC CCCC CCCC CCCC}16
                          TTTT TTTT TTTT TTTT}16                 (6.2.9.2)
```

Remarks 6.2.1.2-6.2.9.2: The ten Eqs. 6.1.1.2-6.2.92 are *not* tautologous, to refute the claimed axioms, denying the ZFC axioms and set theory. The axiom of infinity in Eq. 6.2.5.2 is in fact contradictory regardless of the processing order of terms in the consequent. The book's claimed authority is further eroded by public sources of the axiom of extensionality which make it the only tautologous axiom of the lot.

1 *Axiom of extensionality* If two sets have the same elements then they are identical.

$\forall x \forall y \forall z[(z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$

2 *Null set axiom* There is an empty set, one which contains no elements.

$\exists x \forall y \neg (y \in x)$

It can be shown that there can only be one such set and so it will subsequently be denoted by $\varnothing$.

3 *Pair set axiom* If a and b are sets then there is a set $\{a\}$ whose only element is a and there is a set $\{a, b\}$ whose only elements are a and b.

$\forall x \forall y \exists z \forall w (w \in z \Leftrightarrow w = x \vee w = y)$

4 *Sum set axiom* If a is a set then there is a set $\bigcup a$, the *union* of all the elements of a, whose elements are all the elements of elements of a.

$\forall x \exists y \forall z [z \in y \Leftrightarrow \exists w (w \in a \;\&\; z \in w)]$

5 *Axiom of infinity* There is a set which has $\varnothing$ as an element and which is such that if a is an element of it then $\bigcup\{a, \{a\}\}$ (or $a \cup \{a\}$) is also an element of it.

$$\exists x [\varnothing \in x \;\&\; \forall y (y \in x \Rightarrow \exists z (z \in x \;\&\; \forall w (w \in z \Leftrightarrow w \in y \vee w = y)))]$$

6 *Axiom of foundation* If a is a non-empty set, then there is an element b of a such that there are no sets which belong both to a and b.

$\forall x [\neg (x = \varnothing) \Rightarrow \exists y (y \in x \;\&\; \forall z (z \in x \Rightarrow \neg (z \in y)))]$

7(a) *Subset axiom* If a is a set and $\mathrm{F}(x)$ is any well-formed expression in the language of ZF with a single free variable, then there is a set b whose elements are those elements of a for which $\mathrm{F}(a)$ is true.

$\forall x \exists y \forall z [z \in y \Leftrightarrow z \in x \;\&\; \mathrm{F}(z)]$

7(b) *Replacement axiom* If a is a set and $\mathrm{F}(x, y)$ is a well formed expression in the language of ZF which associates with every element x of a a unique element x^*, then there is a set a^* whose elements are just those sets x^* associated by $\mathrm{F}(x, y)$ with elements of a.

$$\forall x [\forall y [y \in x \Rightarrow \exists z (\mathrm{F}(y, z) \;\&\; \forall w (\mathrm{F}(y, w) \Rightarrow w = z))] \Rightarrow \exists v \forall u [u \in v \Leftrightarrow \exists t (t \in x \;\&\; \mathrm{F}(t, u))]]$$

8 *Power set axiom* If a is a set, then there is a set $\mathrm{P}(a)$, the *power set* of a, whose elements are all the subsets of a.

$\forall x \exists y \forall z [z \in y \Leftrightarrow \forall w (w \in z \Rightarrow w \in x)]$

9 *Axiom of choice* If a is a set, all of whose elements are non-empty sets no two of which have any elements in common, then there is a set c which has precisely one element in common with each element of a.

$$\forall x [\forall y (y \in x \Rightarrow \neg (y = \varnothing)) \;\&\; \forall y \forall z (y \in x \;\&\; z \in x\; (y = z) \Rightarrow \neg (\exists w (w \in y \;\&\; w \in z))) \Rightarrow \exists u \forall y (y \in x \Rightarrow \exists z (z \in u \;\&\; z \in y \;\&\; \forall w (w \in u \;\&\; w \in y \Rightarrow w = z)))]$$

(6.2.1.1-6.2.9.1)